COMPLEX NUMBERS

1. **Complex Numbers:** The number $a + ib$, where a and b are real numbers, i.e.,
"The order pair (a, b), where a and b are real numbers, is called a complex number."

   If $z = a + ib$, then a is called the real part, denoted by $Re\ (z)$ and b is the imaginary part denoted by $Im\ (z)$ of the complex number $Z = a + ib$.

   A complex number $z = x + iy$, $x, y \in R$, is purely real if $y = 0$ i.e., if $\text{Im}\ (z) = 0$ and purely imaginary if $x = 0$, i.e., $Re\ (z) = 0$.

2. **Equal complex numbers:**

$$\text{Let } z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

$$\text{then, } z_1 = z_2 \text{.i.e., } x_1 + iy_1 = x_2 + iy_2$$

$$\Leftrightarrow x_1 = x_2, y_1 = y_2$$

It should be noted that a complex number $z = x + iy = 0 \Leftrightarrow x = 0, y = 0$.

3. **Need for Complex Numbers:** Consider the equation $x^2 + 1 = 0$ or $x^2 = -1$. There exists no real number x which may satisfy that equation. Thus, this arises need to extend the system of real numbers to a system in which equations of the above type can be solved. Euler was first to introduce the symbol of i for the positive square root of -1. The number $\sqrt{-1}$ denoted by i (read as iota) is called the imaginary number.

   Integral power of i

$$i = \sqrt{-1} \;\therefore\; i^2 = -1$$
$$i^3 = i^2.i = (-1) \times i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$

   A number of the form $x + iy$ where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number. The set of complex number is denoted by the letter C.

   If $z = x + iy$ is a complex number, then x is called the real part of z and we write $\text{Re}(z) = x$, y is called the imaginary part of z and we write $\text{Im}(z) = y$.

If $x = 0$ and $y \neq 0$, the complex numbers reduces to the form iy, which is called a pure imaginary number. If $x \neq 0$ and $y = 0$, then the complex number reduces to form x which is a real number.

4. **Set of Complex Number:** The set of complex numbers is denoted by C,

   Where, $C = \{z : z = a + ib, a, b \in R\}$

   or $C = \{z : z = (a, b), \forall a, b \in R\}$

5. **Zero Complex Number:** If complex number $z = a + ib, a, b \in R$ is said to be zero complex number or zero of C if and only if $a = 0$ and $b = 0$.

6. **Negative of a Complex Number:** The complex number $-z = -a - ib$ is called the negative of the complex number $z = a + ib$ and vice-versa.

7. **Equality of two Complex Numbers:** Two complex numbers are said to be equal if and only if their real and imaginary parts are separately equal, i.e., $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.

8. **Algebra of Complex Numbers:**

   (i) Addition of two complex numbers: If $z_1 = a + ib$ and and $z_2 = c + id$, $(a, b, c, d \in R)$ are two complex numbers,

then, their sum is $z_1 + z_2$ and is defined as $z_1 + z_2 = (a + ib) + (c + id)$

$= (a + c) + i\,(b + d)$

Hence, the sum of two complex numbers is again a complex number.

(ii) Subtraction of two complex numbers: If $z_1 = a + ib$ and $z_2 = c + id$ $(a, b, c, d \in \mathrm{R})$.

then, $z_1 - z_2 = z_1 + (-z_2)$; when $-z_2$ is a negative of z_2.

$= (a + ib) + (-c - id)$

$= (a - c) + i\,(b - d)$

Hence, the difference of two complex numbers is again a complex number.

(iii) Multiplication of two complex numbers: Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers, then their product $z_1\,z_2$ is defined by

$z_1 z_2 = (a + ib)\,(c + id)$

$= (ac - bd) + i\,(bc + ad)$

Hence, the multiplication of two complex numbers is a complex number.

(iv) Division of two complex numbers: If $z_1 = a + ib$ and $z_2 = c + id$ ($z_2 \neq 0$ i.e., $c \neq 0$, $d \neq 0$) be two complex numbers, then there exists complex number $z = x + iy$ and such that

$z_1 = z.z_2$

Then, complex number $z = (x + iy)$ is called the quotient of z_1 and z_2 and $z = \frac{z_1}{z_2}$.

9. Basic Properties of Complex Numbers:

(a) Properties of addition in C.

(i) ***Addition is closed in C:*** If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers, then $z_1 + z_2 = (a + ib)\ (c + id) = (a + c) + i\ (b + d)$ which is a complex number.

(ii) ***Addition is commutative in C :*** If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers, then

$$z_1 + z_2 = z_2 + z_1$$

(iii) ***Addition is associative in C :*** If $z_1 = a + ib$, $z_2 = c + id$, $z_3 = e + if$ are three complex numbers, then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

(iv) ***Existence of additive identity in C :*** If $a + ib$ is complex number, then $(a + ib) + (0 + i0) = (a + 0) + i\ (b + 0)$ $= a + ib$

Thus, the complex number $0 + i0$ is the additive identity in C.

(v) ***Existence of inverse in C :*** For every complex number $a + ib$, there exist a complex number $(-a) + i(-b)$ such that $[a + ib] + [(-a) + i(-b)] = [a - a] + i(b - b) = 0 + i\,0 = 0$
= Additive identity of C.

(b) Properties of multiplication

(i) ***Closure law:*** If z_1 and z_2 are two complex numbers, then z_1z_2 is also a complex number.

(ii) ***Associative law:*** If z_1, z_2, z_3 are any three complex numbers, then $(z_1z_2)z_3 = z_1(z_2z_3)$.

(iii) ***Identity law :*** $1 + i0$ is the multiplication identity of C.

(iv) ***Inverse law:*** For every complex number $z = a + ib$ there exists a complex number $(x + iy)$ such that $(a + ib)(x + iy) = 1 + i0$ = Multiplicative identity of C.

This complex number $x + iy$ is called multiplicative inverse of $(a + ib)$.

Rule to find the multiplicative inverse.

If $z = a + ib$ is a non-zero complex

number, then multiplicative inverse of z

is $\frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{z}{|z|^2}$

(v) ***Commutative law:*** If z_1 and z_2 are any two complex numbers, then

$$z_1 z_2 = z_2 z_1$$

(vi) ***Distributive law:*** If z_1, z_2, z_3 are any three complex numbers, then

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

10. Modulus and Argument of a Complex Number: Let $z = x + iy$, then the modulus of z is the positive real number $\sqrt{(x^2+y^2)}$ and is denoted by $|z|$. *i.e.,*

$$|z| = \sqrt{(x^2+y^2)}$$

The argument or amplitude of a complex number $z = x + iy \neq 0$ is any one of the numbers which are solution of the system of equations.

$\cos\theta = \frac{x}{\sqrt{(x^2+y^2)}}$; $\sin\theta = \frac{y}{\sqrt{(x^2+y^2)}}$ is denoted by arg(z) or Arg(z)

The argument of a complex number is not unique. If θ is a value of the argument, then $2n\pi + \theta$, where $n \in I$, are also values of the arguments of z.

Principal value of the argument of the complex numbers z is the argument θ, which satisfy the inequality $-\pi < \theta \leq \pi$.

Note that the argument of the complex number 0 is not defined.

11. Properties of Argument:

(i) The argument of the product of any finite number of complex numbers is equal to sum of their arguments.

$$\therefore \arg(z_1.z_2.z_3 \ldots\ldots z_n) = \theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n$$

$$= \arg z_1 + \arg z_2 + \ldots\ldots + \arg z_n$$

(ii) If z_1 and z_2 are two complex numbers, then $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$$= 2\left[\,|z_1|^2 + |z_2|^2\right]$$

(iii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

12. Conjugate of a Complex Number: If $z = x + iy$ is a complex number, then the complex number $x - iy$ denoted by $\bar{z}$ is called the conjugate of the complex number z, *i.e.*, $\bar{z} = x - iy$.

Properties of Conjugate and Moduli:

(*i*) $|z| = \bar{z}$

(*ii*) $|z^2| = z.\bar{z}$

(*iii*) $\left(\overline{z_1 \pm z_2}\right) = \overline{z_1} \pm \overline{z_2}$

(*iv*) $|z|^2 = z.\bar{z}$

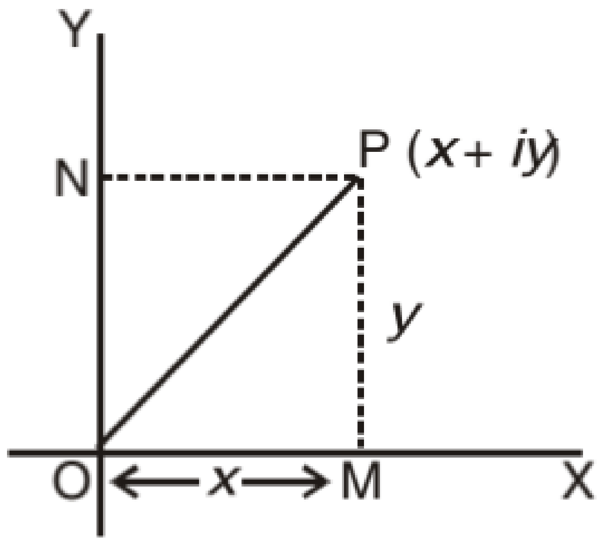
(*v*) $\left(\overline{z_1 z_2}\right) = \bar{z}_1 . \bar{z}_2$

(*vi*) $\left(\overline{z_1 / z_2}\right) = \bar{z}_1 / \bar{z}_2$

(*vii*) $|z_1 + z_2| \leq |z_1| + |z_2|$

(*viii*) $|z_1 - z_2| \geq |z_1| - |z_2|$

13. **Geometrical Representation of Complex Number:** A complex number $z = x + iy$ can be represented by a point P whose cartesian coordinates are (x, y) referred to the axes OX and OY, called the real and imaginary axes respectively.



The plane in which we represent the complex numbers is called the Argand plane or Argand diagram or Complex plane or Gaussian plane.

14. Polar Representation of a Complex Number:

(i) The complex number $z = x + iy$, $x, y \in R$ can be represented in polar form as $z = r(\cos\theta + i\sin\theta)$; where $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{(x^2 + y^2)}$ $= |z|$ and θ, which is arg z, is the solution of the equations.

$$\cos\theta = x/\sqrt{(x^2 + y^2)} = \frac{x}{r} \text{ and}$$

$$\sin\theta = y\Big/\sqrt{\left(x^2+y^2\right)} = \frac{y}{r}$$

(ii) If $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$; then $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Now $z_1 z_2 = r_1 r_2 \cos(\theta_1 + \theta_2)$

$\therefore \; |z_1 z_2| = r_1 r_2$

$\arg(z_1 z_2) = \theta_1 + \theta_2$

Thus, we have $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

Also, $(z_1/z_2) = (r_1/r_2)\cos(\theta_1 - \theta_2)$

$\therefore \; |z_1/z_2| = (r_1/r_2)$ and $\log(z_1/z_2) = \theta_1 - \theta_2$

Hence, we have $|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$.

15. Cube Roots of Unity:

The cube roots of unity are given by 1,

$$\omega = \frac{-1+\sqrt{3}i}{2}, \; \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

Properties of Cube roots of unity:

(i) The sum of the three cube roots of unity is zero *i.e.*, $1 + \omega + \omega^2 = 0$

(ii) The product of the cube roots of unity is one

i.e., $1.\omega.\omega^2 = 1$

i.e., $\omega^3 = 1$

(iii) Each complex cube root of unity is the square of the other.

(iv) Each complex cube root of unity is the reciprocal of the other.

If $\alpha = \frac{-1+\sqrt{3}}{2}$ and

$\beta = \frac{-1-\sqrt{3}}{2}$

Then, $\alpha = \frac{1}{\beta}$ and $\beta = \frac{1}{\alpha}$

16. *n*th Roots of Unity:

(i) The n, nth roots of unity are $\alpha_r = \cos(2\pi r/n) + i \sin(2\pi r/n) = e^{(2\pi r/n)i}$, where $r = 0, 1, 2, \ldots, (n-1)$.

(ii) If α is one of the nth root of unity, then $\alpha^n = 1$.

(iii) The sum of n, nth roots of unity is zero, i.e., $1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = 0$

(iv) The product of n, nth roots of unity is $(-1)^{n-1}$, i.e., $1.\alpha_1.\alpha_2 \ldots\ldots \alpha_{n-1} = (-1)^{n-1}$

17. **Distance between two points:** If z_1, z_2 are two complex numbers then the distance between z_1 and z_2 is $|z_1 - z_2|$.

18. **Point dividing a line segment in a given ratio:** Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be the affixes of the points A and B respectively in the argand plane. If λ be a real number $\neq -1$, then there is a unique point C on AB such that AC : CB = λ : 1.

The point C is given by $\dfrac{x_1 + \lambda x_2}{\lambda + 1}, \dfrac{y_1 + \lambda y_2}{\lambda + 1}$.

The affix of C is therefore, $(z_1 + \lambda z_2)/(1 + \lambda)$

Remark: The affix of the mid-point of z_1, z_2 is $(z_1 + z_2)/2$.

19. If z_1, z_2, z_3 be the affixes of the vertices of a triangle, the centroid of the triangle has the affix $(z_1 + z_2 + z_3)/3$.

20. Three points z_1, z_2, z_3 are collinear if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

21. De-moivre's Theorem: If n is any integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta = \cos(-n\theta) + i\sin(-n\theta)$$

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = \cos(-n\theta) + i\sin(-n\theta)$$

22. Some Applications of De-Moivre's theorem

(*i*) *To express cos nθ and sin nθ in powers of cosθ and sinθ:*

$$(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n = \cos^n\theta + {}^nC_1 \cos^{n-1}\theta . i\sin\theta + {}^nC_2 \cos^{n-2}\theta . i^2\sin^2\theta + ...$$

Equating real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - .....$$

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 . \cos^{n-3}\theta \sin^3\theta + ...$$

(*ii*) *To find roots of a complex number:*

Let $z = x + iy$, and we have to find its

nth root, putting z in polar form, we get

$$z = r(\cos\theta + i\sin\theta);\ r = |z| = \sqrt{(x^2 + y^2)},$$

and θ is the principal arg of z. Then considering general value of arg z, we have

$$z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)],\ \text{then}$$

$$z^{1/n} = r^{1/n}\left[\cos\frac{(2k\pi + \theta)}{n} + i\sin\frac{(2k\pi + \theta)}{n}\right] \quad ....(i)$$

$$k = 0, 1, 2, ..., (n-1)$$

Equation (i) gives n distinct roots of z. These roots can be written as

$$z^{1/n} = r^{1/n}\, e(2^{k\pi + \theta})^{i/n};$$

$$k = 0, 1, 2, ... (n-1) \text{ or}$$

$$z^{1/n} = r^{1/n}.\, e^{2k\pi i/n}.\, e^{\theta i/n}$$

which shows that the roots are in G.P. of first term $r^{1/n}\, e^{\theta i/n}$ and common ratio $e^{2\pi i/n}$. We exhibit the process by giving an example.

Find the fourth roots of $1 + i\sqrt{3}$

Sol. $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\sqrt{3}/2\right)$

$= 2\ [\cos \pi/3 + i \sin \pi/3]$

$= 2\ [\cos (2n\pi + \pi/3) + i \sin (2n\pi + \pi/3)]$

$\therefore \quad (1 + i\sqrt{3})^{1/4} = 2^{1/4}\left[\cos\left(\frac{6n\pi + \pi}{12}\right) + i\sin\left(\frac{6n\pi + n}{12}\right)\right];$

$n = 0, 1, 2, 3.$